

Double Averaging Approach to the Study of Spinup Dynamics of Flexible Satellites

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We study the dynamics of a class of flexible dual-spin satellites in which the rotor is spun up by a small constant torque T applied by the platform. We use a previously published zero torque solution that was obtained using the Krylov–Bogoliubov–Mitropolski averaging method. A second application of averaging developed herein leads to a reduction of the equations of motion from a sixth-order system to a single first-order equation describing the slow evolution of energy as a function of the axial angular momentum of the rotor. The geometrical interpretation of this reduction involves projecting solutions onto a certain bifurcation diagram. Numerical solutions of the averaged equation agree with numerical solutions to the full sixth-order system and show that the flexible spacecraft behaves essentially the same as its rigid counterpart during the spinup maneuver.

Introduction

WE are interested in studying the attitude dynamics of dual-spin spacecraft with one or more flexible appendages during the spinup maneuver. In this paper we give a limited treatment of this problem for the case in which the spacecraft is nominally axisymmetric about the rotor's spin axis and the spacecraft is free of external forces and torques. The dynamics problem of interest can be considered as a sequence of perturbations from a known integrable problem. Specifically, when both the rotor and platform are axisymmetric, and the axis of relative spin coincides with their axes of symmetry, and neither axial torques nor flexibility are included, then the problem is essentially the same as the axisymmetric rigid body with body-fixed torque. In this case, the problem is trivially integrable. This is the simplest example of a gyrostat and is usually referred to as an axial, axisymmetric gyrostat.

The perturbations result from one or more of the following: platform asymmetry, rotor misalignment, spinup torque, and flexibility. Addition of any one of the first three perturbations leaves the system integrable. For example, allowing platform asymmetry or rotor misalignment (or both) results in an integrable gyrostat problem that was first solved by Volterra.¹ The addition of a spinup torque for axial axisymmetric gyrostats was first solved by Sen and Bainum.² Spinup of gyrostats with platform asymmetry and rotor aligned with a principal axis was studied by Gebman and Mingori.³ They obtained an approximate solution for the specific case of flat spin recovery. The general problem of spinup of axial gyrostats with platform asymmetry was studied by Hall and Rand,⁴ and spinup of more general gyrostats was studied by Hall.^{5,6} The addition of flexible appendages to gyrostats has been studied by numerous authors; a comprehensive review of early work was done by Modi.⁷ The model used in this paper consists of a rigid platform and rotor with a flexible appendage whose vibrations are constrained to purely torsional modes (e.g., constrained by guy wires). This model was first studied by Mazzoleni and Schlack⁸ for the purpose of determining stability

boundaries in a gravity gradient field. An approximate solution to the equations of motion for the torque-free case was derived via the method of averaging by Stabb and Schlack.⁹

In this paper we study via a double averaging approach the dynamics of a class of flexible dual-spin satellites when the rotor is spun up by a small constant torque T applied by the platform. Our approach is based on the approximate zero torque solution developed in Ref. 9 and uses the averaging approach developed for small torque in Ref. 4. Specifically, we apply averaging as in Ref. 4, wherein the slow evolution of energy is obtained as a function of the angular momentum of the rotor about its spin axis. This requires a known periodic $T = 0$ solution that is taken from Ref. 9, wherein the Krylov–Bogoliubov–Mitropolski (KBM) averaging method¹⁰ is applied to a dual-spin spacecraft with a flexible appendage and zero spinup torque applied to the rotor. Note that the method of averaging is applied twice in this process, first in order to obtain the $T = 0$ solution from Ref. 9 and then to obtain the slow evolution of energy (as in this work). In each case, a small parameter is available to justify the use of the averaging approximation. In Ref. 9 this small parameter is ε , whereas in this work it is the torque T .

It has been noted that ad hoc application of the method of averaging may lead to erroneous results.¹¹ On the other hand, the method of averaging is essentially the only perturbation method for which a relevant theorem is available to establish the accuracy of the averaged equations with respect to the exact equations (e.g., theorem 3.2.10 of Ref. 12). Both applications of the method of averaging used herein are carried out in accordance with the averaging theorem; hence the results of each averaging process are accurate as the respective small parameter tends to zero. We must point out, however, that in the second application of averaging developed in this paper, some additional approximations are required in order to obtain analytical results. These approximations are justified and pointed out as they are made.

We sketch the approach here and give complete details in the remainder of the paper. The essential quantity of interest is the energy or Hamiltonian H that is constant for zero spinup torque ($T = 0$). Let I_r and ω_r represent the axial moment of inertia and angular velocity of the rotor. The rotor angular velocity satisfies

$$\dot{\omega}_r = T/I_r \quad (1)$$

It is shown that for any spinup torque T , the Hamiltonian satisfies

$$\dot{H} = T f \quad (2)$$

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where f is a function of appendage deflection and the angular velocities. Note that f is quite complicated [cf. Eq. (29) presented subsequently]; however, it has the nice property that it is a periodic function of time when $T = 0$. Furthermore, an approximation for f can be computed explicitly using the $T = 0$ solution obtained in Ref. 9, making it possible to apply averaging to Eq. (2).

Since f is periodic (with period P) and T is a small constant, Eq. (2) is in the correct form for averaging,¹² which leads to

$$\dot{\bar{H}} = T \bar{f} \quad (3)$$

where

$$\bar{f} \triangleq \frac{1}{P} \int_0^P f dt \quad (4)$$

and \bar{H} is an approximation to H . Note that \bar{f} is not explicitly a function of time, since the time dependence is removed by averaging. Some additional approximations are required in order to evaluate the integral in Eq. (4). It is shown that Eqs. (1) and (3) form an approximate autonomous system of equations for H and ω_r .

Finally, we eliminate t from Eqs. (1) and (3), obtaining

$$\bar{H}' = I_r \bar{f} \quad (5)$$

where $(\cdot)' \triangleq d(\cdot)/d\omega_r$. In this way, we reduce the system of differential equations of motion from a sixth-order system to a single first-order equation. The averaged equation is studied numerically, and solutions are compared to solutions of the exact system.

Problem Formulation and Equations of Motion

Satellite Model

The model to be examined is depicted in Fig. 1. It is composed of a main body or platform, upon which is mounted a rotor that spins relative to the platform. In addition to the rotor, a flexible appendage is cantilevered off the main body. The points c_i are the centers of mass for each of the components, and c_m is the center of mass of the system. The rotor is considered part of the main body for the purpose of determining the center of mass. Figure 2 shows the body in a deformed state, with tip rotation about the beam axis. This is the only displacement mode to be considered as might happen if the appendage were constrained by guy wires.⁸

The following assumptions are made in this model.

1) A set of coordinate axes is attached to the center of mass of the platform and is parallel to the principal axes in the undeformed

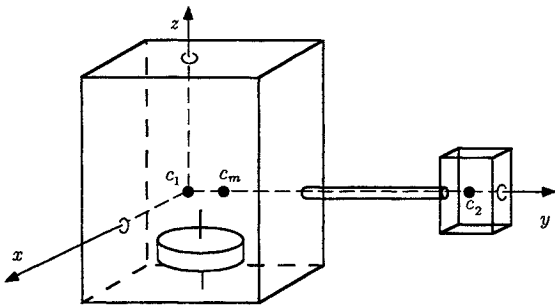


Fig. 1 Dual-spin satellite with flexible appendage.

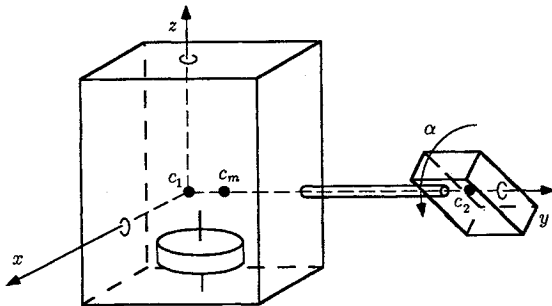


Fig. 2 Satellite in deformed configuration.

state. The center of mass of the tip mass, and hence of the system, lies on the y axis.

2) In the undeformed state, the principal moments of inertia of the system about the x and y axes are equal, the moment of inertia about the z axis is arbitrary, and the tip mass has its principal axes aligned with the system's axes.

3) A symmetric, rigid rotor is aligned with the z axis.

4) The system is force and torque free.

5) The beam is massless.

6) The tip mass is assumed small compared to the spacecraft as a whole.

The equations of motion are formulated by creating the Lagrangian for the total system and applying the extended Hamilton's principle along with Lagrange's equations for quasi-coordinates.¹⁰ Let $\theta(y, t)$ be the rotation of the beam relative to the xyz axes and assume that the rotation of the continuous beam can be replaced by a series of generalized coordinates in time and admissible functions in space, such as

$$\theta(y, t) = \sum_{i=1}^n \theta_i(t) Y_{\theta i}(y) \quad (6)$$

so the equations can be converted into ordinary differential equations in time only. The rotation that the tip mass experiences is simply Eq. (6) evaluated at $y = L$, where L is the length of the beam. The kinetic energy of the platform and rotor is given by

$$K_1 = \frac{1}{2} [\omega^T I_1 \omega + I_r (\omega_s + \omega_z)^2 + m_1 \dot{\mathbf{r}}^T \dot{\mathbf{r}}] \quad (7)$$

where $\omega = (\omega_x, \omega_y, \omega_z)$ is the angular velocity of the platform, ω_s is the angular velocity of the rotor relative to the platform, I_r is the moment of inertia of the rotor about its spin axis, and I_1 is the inertia tensor of the platform and rotor together about c_1 except about the z axis where it is just for the platform. The velocity of point c_1 is $\dot{\mathbf{r}}_1 = \omega \times \mathbf{r}_1$ where \mathbf{r}_1 is the vector from c_m to c_1 . The rotor's energy from spin about the z axis is given by the second term in Eq. (7). The kinetic energy of the tip mass is given by

$$K_2 = \frac{1}{2} (\omega_2^T I_2 \omega_2 + m_2 \dot{\mathbf{r}}_2^T \dot{\mathbf{r}}_2) \quad (8)$$

where $\omega_2 = \{(\omega_x \cos \theta - \omega_z \sin \theta), [\omega_y + \dot{\theta}(L, t)], (\omega_x \sin \theta + \omega_z \cos \theta)\}$ is the angular velocity of the tip mass expressed in terms of principal coordinates of the tip mass, and $I_2 = \text{diag}[I_x, I_y, I_z]$ is the principal inertia tensor of the tip mass about c_2 . It is also diagonal relative to the xyz frame in the undeformed case. The velocity of point c_2 is $\dot{\mathbf{r}}_2 = \omega \times \mathbf{r}_2$ where \mathbf{r}_2 is the vector from c_m to c_2 . Using the parallel axis theorem it can be shown that the total kinetic energy of the satellite reduces to

$$K = \frac{1}{2} [\omega^T (I_s - I_2) \omega + I_r ((\omega_s + \omega_z)^2 + \omega^T I_2 \omega)] \quad (9)$$

where $I_s = \text{diag}[A, A, C]$ is the moment of inertia tensor for the entire satellite.

The potential energy of the system comes from the elastic beam and is given by

$$V = \frac{1}{2} \int_0^L \left(\frac{\partial \theta}{\partial y} \right)^2 JG dy \quad (10)$$

For the assumption of a massless beam, the use of only one admissible function, y/L , gives an exact representation of the beam's deformation, since the mode shapes for pure torsional vibration of a massless, uniform beam are governed by an equation of the form $\partial^2 \theta / \partial y^2 = 0$. At the tip mass we define $\alpha(t) = \theta(L, t)$. This gives $\theta(y, t) = \alpha(t) y/L$. With this simplification and the introduction of the beam stiffness $K_y = JG/L$, the potential energy can be integrated and simply becomes

$$V = \frac{1}{2} \int_0^L \left(\frac{\alpha(t)}{L} \right)^2 JG dy = \frac{JG \alpha(t)^2}{2L} = \frac{K_y \alpha(t)^2}{2} \quad (11)$$

With the energy now only a function of time, Hamilton's principle reduces to Lagrange's equations. Since the strain energy V is accounted for in the Lagrangian, the generalized forces Q_i are solely

from forces not derivable from a potential function. For this analysis, the Q_i are taken as zero except for the torque on the rotor that will be allowed to be nonzero.

Equations of Motion

The Lagrangian for the system $K - V$ is used in the formulation of the equations of motion using Lagrange's equations. The equations of motion obtained for the system are

$$\begin{aligned} & [A - (I_x - I_z) \sin^2 \alpha] \dot{\omega}_x + [C - A + (I_x - I_z) \sin^2 \alpha] \omega_y \omega_z \\ & - (I_x - I_y) \cos \alpha \sin \alpha (\omega_x \omega_y + \dot{\omega}_z) \\ & - 2(I_x - I_z) \cos \alpha \sin \alpha \omega_x \dot{\alpha} \\ & + [-I_y + (I_x - I_z)(\sin^2 \alpha - \cos^2 \alpha)] \omega_z \dot{\alpha} \\ & + I_r \omega_y (\omega_z + \omega_s) = 0 \end{aligned} \quad (12)$$

$$\begin{aligned} & A \dot{\omega}_y + [A - C - 2(I_x - I_z) \sin^2 \alpha] \omega_x \omega_z \\ & + (I_x - I_z) \cos \alpha \sin \alpha (\omega_x^2 - \omega_z^2) + I_y \ddot{\alpha} \\ & - I_r \omega_x (\omega_z + \omega_s) = 0 \end{aligned} \quad (13)$$

$$\begin{aligned} & [C + (I_x - I_z) \sin^2 \alpha] \dot{\omega}_z + I_r (\dot{\omega}_z + \dot{\omega}_s) + (I_x - I_z) \sin^2 \alpha \omega_x \omega_y \\ & + (I_x - I_z) \cos \alpha \sin \alpha (\omega_y \omega_z - \dot{\omega}_x) \\ & - 2(I_x - I_z) \cos \alpha \sin \alpha \omega_z \dot{\alpha} \\ & + [I_y + (I_x - I_z)(\sin^2 \alpha - \cos^2 \alpha)] \omega_x \dot{\alpha} = 0 \end{aligned} \quad (14)$$

$$I_r (\dot{\omega}_z + \dot{\omega}_s) = T \quad (15)$$

$$\begin{aligned} & I_y \ddot{\alpha} + I_y \dot{\omega}_y + K_y \alpha + (I_x - I_z) (\omega_x \cos \alpha - \omega_z \sin \alpha) \\ & \times (\omega_x \sin \alpha + \omega_z \cos \alpha) = 0 \end{aligned} \quad (16)$$

where T is the torque on the rotor from the drive motor. Note that this is a sixth-order system of equations. Although simulation results are useful, it is possible to miss important nonlinear phenomena when conclusions are based on simulation alone.¹³

Slow Evolution of Energy vs Rotor Angular Momentum for Rigid Gyrostat

To relate the present work to that of Ref. 4, for now we specialize Eqs. (12)–(16) for a rigid gyrostat by letting $\alpha = \dot{\alpha} = 0$. This will motivate the more complicated development for the flexible case, which follows. With $\alpha = \dot{\alpha} = 0$, the equations of motion become

$$\dot{\omega}_x = [(A - C)/A] \omega_y \omega_z - (I_r \omega_r / A) \omega_y \quad (17)$$

$$\dot{\omega}_y = [(C - A)/A] \omega_x \omega_z + (I_r \omega_r / A) \omega_x \quad (18)$$

$$\dot{\omega}_z = -I_r \dot{\omega}_r / C \quad (19)$$

$$\dot{\omega}_r = T / I_r \quad (20)$$

These equations describe the rotational motion of an axial, axisymmetric gyrostat. The Hamiltonian in this case is simply the rotational kinetic energy that is constant for $T = 0$ and is given by

$$K = \frac{1}{2} (\omega^T I_s \omega + I_r \omega_r^2) \quad (21)$$

When $T \neq 0$, H satisfies

$$\dot{H} = \frac{\partial H}{\partial \omega_x} \dot{\omega}_x + \frac{\partial H}{\partial \omega_y} \dot{\omega}_y + \frac{\partial H}{\partial \omega_z} \dot{\omega}_z + \frac{\partial H}{\partial \omega_r} \dot{\omega}_r \quad (22)$$

which simplifies to

$$\dot{H} = T(\omega_r - \omega_z) = T \omega_s \quad (23)$$

In the unperturbed problem ($T = 0$), both ω_r and ω_z are constants; hence, ω_s is constant, and \dot{H} is trivially integrable. (Note

that this is a result of the gyrostat's axisymmetry and is not true for an asymmetric gyrostat.) This implies that the equations for ω_x and ω_y form a constant coefficient linear system that is integrable and gives the transverse angular velocities in terms of trigonometric functions. The simplicity of this case would allow a more straightforward treatment than we give; we proceed formally, however, as this section is intended to motivate the following section where flexibility is included. Thus, we treat the right-hand side of \dot{H} as a periodic function of time with period P . Then Eqs. (23) and (20) are in an appropriate form for applying the method of averaging.¹² We use overbars to represent averaged approximations:

$$\bar{(\cdot)} = \frac{1}{P} \int_0^P (\cdot) dt \quad (24)$$

Equations (23) and (20) become

$$\dot{\bar{H}} = T \bar{\omega}_s \quad (25)$$

$$\dot{\omega}_r = T / I_r \quad (26)$$

Note that here the equations are actually unchanged by averaging, since ω_s is constant. In the case of asymmetry and/or flexibility, however, \bar{H} will be an approximation to \bar{H} ; in either case, $\dot{\omega}_r$ will still be exact, since we assume constant spinup torque.

As the averaged equations are autonomous, we can eliminate t , obtaining

$$\bar{H}' = I_r \bar{\omega}_s = I_r (\overline{\omega_r - \omega_z}) \quad (27)$$

where $(\cdot)' = d(\cdot)/d\omega_r$. Since Eq. (19) implies $\omega_z = \omega_z(\omega_r)$, this defines $\bar{H} = \bar{H}(\omega_r) = \bar{H}(\omega_r)$. Since averaging is accurate to $\mathcal{O}(T)$ on time scales of $\mathcal{O}(1/T)$, we expect the approximate $\bar{H}(\omega_r)$ to be close to the exact solution obtained by numerically integrating the exact equations and calculating H . In this simple case we can integrate Eq. (27) exactly, which gives \bar{H} as a quadratic function of ω_r . This is, in fact, an exact solution for $H(\omega_r)$, as may be checked by direct calculation. In the case of an asymmetric gyrostat, or an axisymmetric gyrostat with flexibility, only approximate results are possible.

Given the slow equations defining $\bar{H}(\omega_r)$, our approach follows Ref. 4, where the $\omega_r H$ plane is used to study spinup dynamics for asymmetric rigid gyrostats. (Actually Ref. 4 uses rotor momentum rather than angular velocity and an energylike quantity y that is similar to H used here.) First, we compute the equilibrium values of Eqs. (17)–(19) for $T = 0$, with ω_r as a parameter. Then, we compute the equilibrium values of H along these branches that define curves in the $\omega_r H$ plane, denoted $H_e(\omega_r)$. An example $\omega_r H$ plane is shown in Fig. 3. The curves represent the value of the Hamiltonian as a function of ω_r for equilibrium points of Eqs. (17)–(19). The two branches emanating from $H \approx 30$, $\omega_r = 0$ correspond to spin about the z axis: the lower curve is the usual dual-spin equilibrium, whereas the upper curve is for a counter-rotating platform. The branch starting at $H \approx 50$ corresponds to spin about a transverse axis. Note that this branch of equilibria intersects the counter-rotating branch at $\omega_r \approx 0.8$, $H \approx 150$. This pitchfork bifurcation is discussed further in Ref. 4.

The $\omega_r H$ plane serves two purposes in studying gyrostat dynamics: it is a useful bifurcation diagram for studying the change in equilibria as the rotor velocity is changed as a parameter, and it represents the slow state space for the system, since H and ω_r are the slowly varying quantities described by Eqs. (23) and (20). Trajectories of $\bar{H}(\omega_r)$ can be compared with exact solutions for $H(\omega_r)$, further justifying this approach. Unfortunately, for the axisymmetric rigid gyrostat, the $\omega_r H$ plane is not very interesting in that all of the branches of equilibria are stable. In the asymmetric case, there exist unstable branches and bifurcation points where unstable and stable branches intersect. Thus, the $\omega_r H$ plane is useful for identifying spinup trajectories that may pass near unstable equilibria of the unperturbed system, typically resulting in large nutation growth. The present application illustrates the accuracy of the averaged vs exact equations for spinup of a flexible satellite.

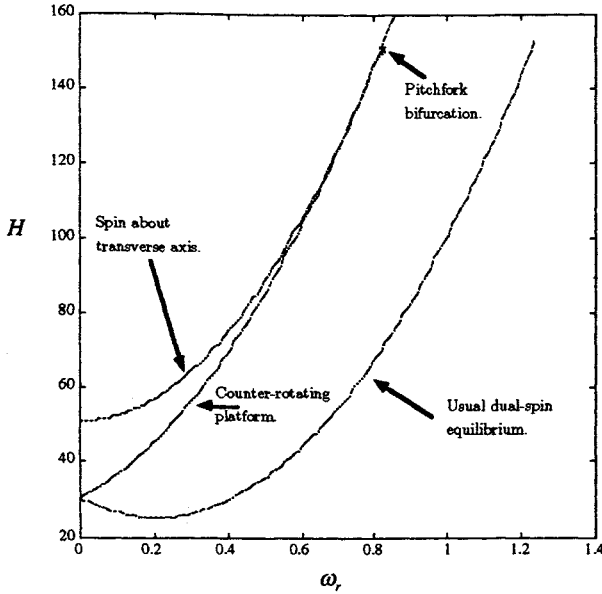


Fig. 3 Equilibria for rigid gyrost.

Slow Evolution of Energy vs Rotor Angular Momentum for Gyrost with Flexible Appendage

Derivation of Averaged Equation

For the flexible gyrost considered in this paper, the Hamiltonian includes the potential energy term in Eq. (11). The differential equation describing H is again obtained by the chain rule

$$\dot{H} = \frac{\partial H}{\partial \omega_x} \dot{\omega}_x + \frac{\partial H}{\partial \omega_y} \dot{\omega}_y + \frac{\partial H}{\partial \omega_z} \dot{\omega}_z + \frac{\partial H}{\partial \omega_r} \dot{\omega}_r + \frac{\partial H}{\partial \alpha} \dot{\alpha} + \frac{\partial H}{\partial \dot{\alpha}} \dot{\ddot{\alpha}} \quad (28)$$

that simplifies to

$$\dot{H} = T \left(\frac{1}{I_r} \frac{\partial H}{\partial \omega_r} - \frac{1}{[C + (I_x - I_z) \sin^2 \alpha]} \frac{\partial H}{\partial \omega_z} \right) \quad (29)$$

$$\frac{d\bar{H}}{d\omega_r} = \frac{1}{P} \int_0^P \left(I_r \omega_r - \frac{I_r (h^2/2I_r \omega_r - I_r \omega_r/2)}{\{C + (I_x - I_z) \sin^2 [\sqrt{(2\bar{H} - I_r \omega_r^2)/K_y} \cos(\sqrt{K_y/I_y} t)]\}} \right) dt \quad (39)$$

Applying averaging as in the preceding section yields

$$\dot{\bar{H}} = \frac{T}{P} \int_0^P \left(\frac{1}{I_r} \frac{\partial H}{\partial \omega_r} - \frac{1}{[C + (I_x - I_z) \sin^2 \alpha]} \frac{\partial H}{\partial \omega_z} \right) dt \quad (30)$$

Dividing $\dot{\bar{H}}$ by $\dot{\omega}_r$, we obtain

$$\begin{aligned} \frac{d\bar{H}}{d\omega_r} &= \frac{\dot{\bar{H}}}{\dot{\omega}_r} = \frac{I_r \dot{\bar{H}}}{I_r \dot{\omega}_r} = \frac{I_r \dot{\bar{H}}}{T} \\ &= \frac{1}{P} \int_0^P \left(\frac{\partial H}{\partial \omega_r} - \frac{I_r}{[C + (I_x - I_z) \sin^2 \alpha]} \frac{\partial H}{\partial \omega_z} \right) dt \quad (31) \end{aligned}$$

The integrand in Eq. (31) may be approximated by using the approximate $T = 0$ solutions developed in Ref. 9. Then the additional assumption is made that the torsional motion of the flexible appendage remains small, from which the integral can be computed exactly. The partial derivative $\partial H/\partial \omega_r$ is given by

$$\frac{\partial H}{\partial \omega_r} = I_r \omega_r \quad (32)$$

and using Eqs. (27–30) from Ref. 9, $\partial H/\partial \omega_z$ can be approximated by

$$\frac{\partial H}{\partial \omega_z} = \frac{h^2}{2I_r \omega_r} - \frac{I_r \omega_r}{2} + \mathcal{O}(\varepsilon^2) \approx \frac{h^2}{2I_r \omega_r} - \frac{I_r \omega_r}{2} \quad (33)$$

where h is the magnitude of the total angular momentum of the spacecraft and ε is the small parameter used in obtaining the perturbation solution for $T = 0$. The small parameter ε is introduced to exploit the assumption that the tip mass is small compared to the platform mass; specifically, the ratio of the tip mass moments of inertia to the platform moments of inertia are $\mathcal{O}(\varepsilon)$ (see Refs. 9 and 14).

From Eqs. (14), (30), and (32) of Ref. 9 we can make the following approximations:

$$\sin \alpha = \sin[e \cos \psi + \varepsilon A(a, b, e, \xi, \psi)] \approx \sin(e \cos \psi) \quad (34)$$

$$\cos \psi = \cos(\gamma \tau + \varepsilon \psi_1 \tau) \approx \cos(\gamma \tau) = \cos(\sqrt{K_y/I_y} t) \quad (35)$$

where e is the amplitude in radians of the torsional vibration of the flexible appendage. Using Eqs. (27–30) of Ref. 9 the Hamiltonian may be expressed as

$$H = \frac{I_r \omega_r^2}{2} + \frac{K_y e^2}{2} + \mathcal{O}(\varepsilon^2) \approx \frac{I_r \omega_r^2}{2} + \frac{K_y e^2}{2} \quad (36)$$

Solving this expression for e , we obtain

$$e = \sqrt{\frac{2H - I_r \omega_r^2}{K_y}} \quad (37)$$

from which $\sin \alpha$ may be approximated by

$$\sin \alpha \approx \sin \left[\sqrt{\frac{2H - I_r \omega_r^2}{K_y}} \cos \left(\sqrt{\frac{K_y}{I_y}} t \right) \right] \quad (38)$$

Thus, we obtain the following averaged, approximate equation:

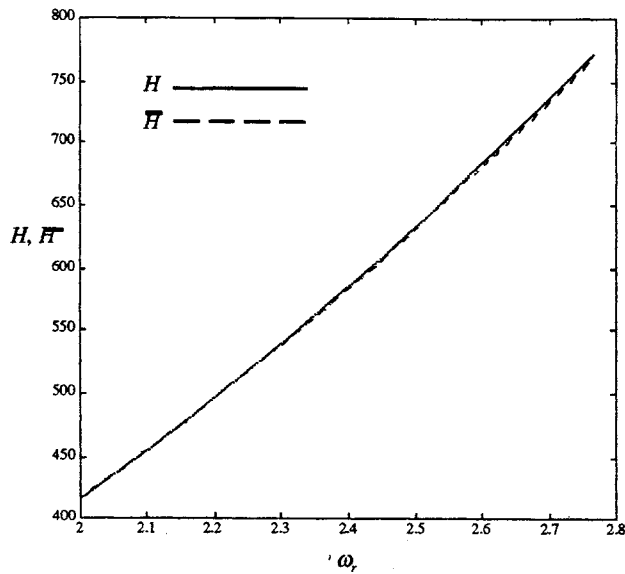
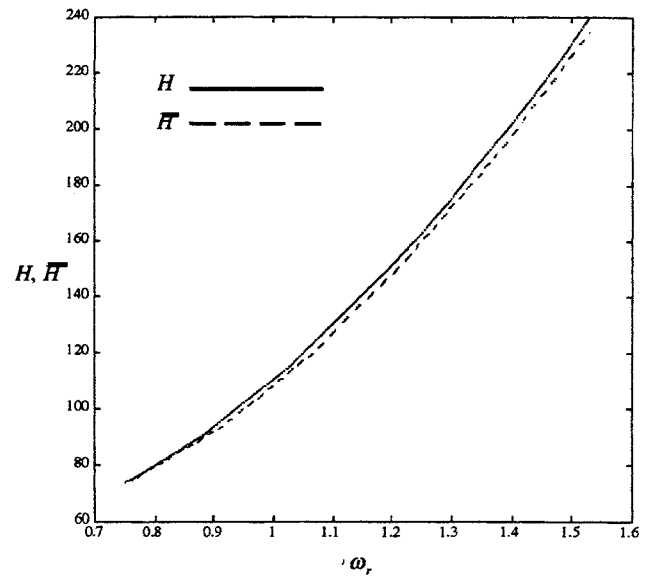
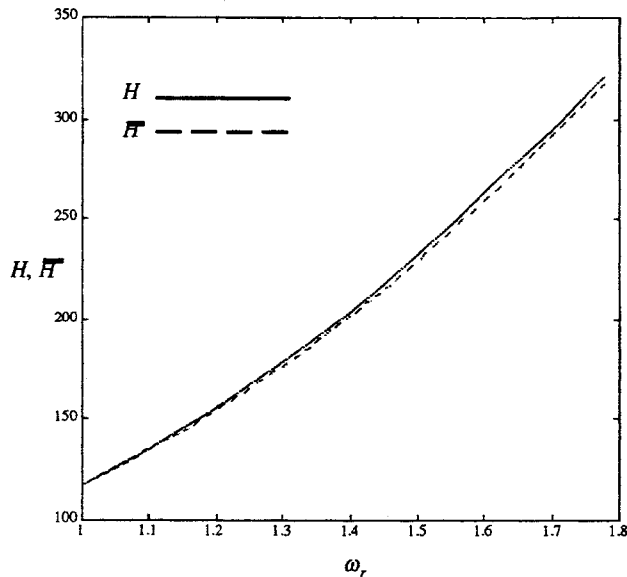
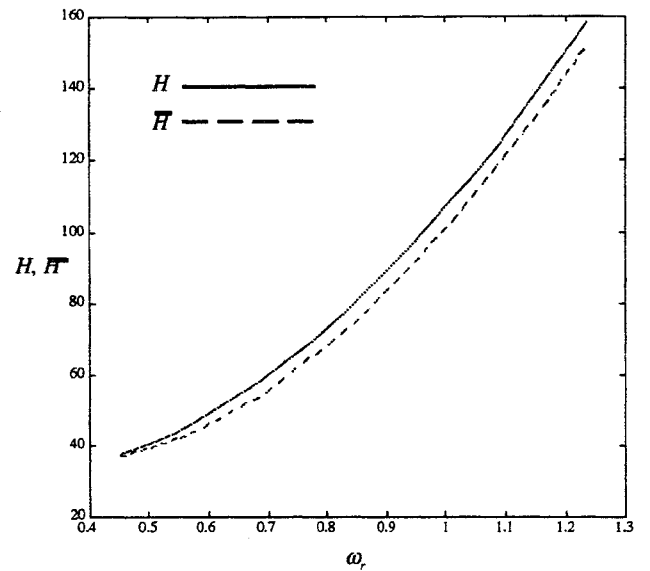
Note from Eq. (39) that the period of the integrand is $P = \pi \sqrt{I_y/K_y}$. The integral in Eq. (39) is apparently nonelementary. We can, however, obtain an approximate expression for the right-hand side by noting that e is small, so that we can expand $\sin^2(e \cos \psi)$ as a Taylor series for small e . This gives

$$\sin^2(e \cos \psi) = e^2 \cos^2 \psi - e^4 \cos^4 \psi/3 + \mathcal{O}(e^6) \quad (40)$$

Substituting Eq. (40) and Eqs. (32–37) into Eq. (31) allows us to evaluate the integral in Eq. (31) term by term, thereby obtaining the following approximate equation for $d\bar{H}/d\omega_r$:

$$\begin{aligned} \frac{d\bar{H}}{d\omega_r} &= I_r \left(1 + \frac{I_r}{2C} \right) \omega_r - \frac{h^2}{2C \omega_r} \\ &\quad + \frac{(I_x - I_z)(2\bar{H} - I_r \omega_r^2)(h^2 - I_r^2 \omega_r^2)}{4C^2 K_y \omega_r} \quad (41) \end{aligned}$$

This single first-order differential equation approximately describes the motion of the spacecraft during the spinup maneuver. In the next section we demonstrate the quality of the approximation, as well as discuss the limitations. It should be noted that Eq. (41) is nonlinear and is not separable, thus no analytical solution is evident. Numerical integration of the averaged equation, however, is simpler than for the full system and is independent of T . As the spinup torque T

Fig. 4 H and \bar{H} vs ω_r for $\omega_{r0} = 2$.Fig. 6 H and \bar{H} vs ω_r for $\omega_{r0} = 0.75$.Fig. 5 H and \bar{H} vs ω_r for $\omega_{r0} = 1$.Fig. 7 H and \bar{H} vs ω_r for $\omega_{r0} = 0.45$.

is decreased, the time to complete the spinup maneuver and, hence, the time to integrate the full equations are increased. By contrast, since Eq. (41) is independent of T , the required integration time is also independent of T , making it less expensive to integrate for small T . Since averaging is based on a small T assumption, the averaged equation increases in accuracy and cost effectiveness as T is decreased.

Comparison of Averaged Equation with Direct Integration of Exact Equations

We now compare plots of \bar{H} vs ω_r obtained via the averaged Eq. (41) with plots of H vs ω_r obtained by numerically integrating the equations of motion [Eqs. (12–16)] for $\omega_x, \omega_y, \omega_z, \alpha, \dot{\alpha}$ and calculating $H = K + V$. The plots can be compared by noting that Eq. (15) implies $\omega_r = \omega_r(0) + Tt/I_r$.

The parameter values and initial conditions chosen for the comparison are

$A = 600$	$C = 1000$	$I_r = 200$	$I_x = 26.5$
$I_y = 21.2$	$I_z = 21.2$	$\omega_x(0) = 0.1$	$\omega_y(0) = 0$
$\omega_z(0) = 0.15$	$\alpha(0) = 0.1$	$\dot{\alpha}(0) = 0$	$K_y = 523$

We plot \bar{H} and H vs ω_r for various values of $\omega_r(0)$ as summarized in Table 1 and shown in Figs. 4–8. We choose $T = I_r$ and spin the

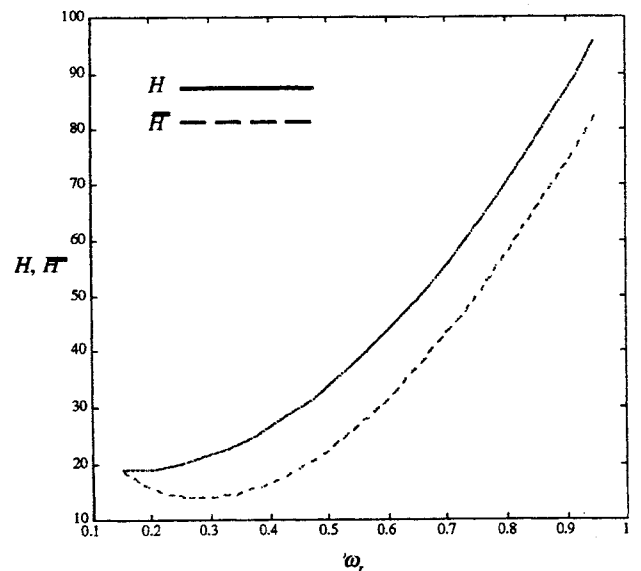
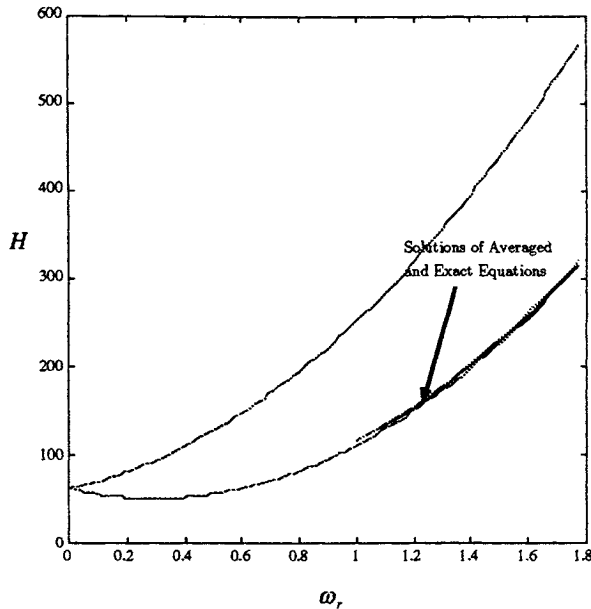
Fig. 8 H and \bar{H} vs ω_r for $\omega_{r0} = 0.15$.

Table 1 Values of $\omega_r(0)$ for plots

Figure	4	5	6	7	8
$\omega_r(0)$	2	1	0.75	0.45	0.15

Fig. 9 Overlay of solutions on equilibrium plots for $\omega_{r0} = 0$.

rotor until $(I_r \omega_r)^2 = h^2$. From the plots we see that there is excellent agreement between the averaged and exact solutions, provided $\omega_r(0)$ is not too close to $\omega_z(0)$. This is expected since our $T = 0$ solution was derived assuming $\omega_r(0)$ was not close to $\omega_z(0)$.

Not only does \bar{H} compare well with H , but as seen in Fig. 9, both solutions follow the equilibrium branch $H_e(\omega_r)$ in the $\omega_r H$ plane. Thus the solution remains close to the unperturbed equilibrium solution branch for a rigid axisymmetric gyrostat. As noted in the preceding section, all of the equilibrium solution branches in the $\omega_r H$ plane are stable in this case. The two main branches also correspond to equilibria of the full equations (with flexibility); however, the transverse spin branch is not an equilibrium branch and, hence, is not shown in the figure. We have not calculated the equivalent of this branch for the flexible case. It is possible that such a branch might be unstable for some values of ω_r that would indicate an unsafe region for a spinup trajectory to pass through.

Conclusions

We have demonstrated the feasibility of using double averaging to study spinup dynamics of flexible dual-spin satellites by reducing

the sixth-order system of differential equations describing the motion to a single approximate first-order equation. Reduction of the dynamics to a single equation allows the projection of solutions onto a bifurcation diagram in the plane whose coordinates are the rotor angular momentum and the system Hamiltonian. The results obtained by averaging closely match results obtained by integrating the equations of motion directly and show that the flexible dual-spin spacecraft studied here behaves similarly to the rigid spacecraft during the spinup maneuver.

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